Intrinsic semiharmonic maps

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Abstract

For maps from a domain $\Omega \subset \mathbb{R}^m$ into a Riemannian manifold N, a functional coming from the norm of a fractional Sobolev space has recently been studied by Da Lio and Rivière. An intrinsically defined functional with a similar behaviour also exists, and its first variation can be identified with a Dirichlet-to-Neumann map belonging to the harmonic map problem. The critical points have regularity properties analogous to harmonic maps.

Key words: Harmonic map, Dirichlet-to-Neumann map, regularity
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Harmonic maps between Riemannian manifolds are particularly interesting on a two-dimensional domain, because in this case, the problem is invariant under conformal transformations. Using this fact as a motivation, Da Lio and Rivière [2] proposed to study a functional for maps on a one-dimensional domain with related properties. This functional is given in terms of the seminorm belonging to the homogeneous fractional Sobolev space $\dot{H}^{1/2}(\mathbb{R})$. Such a quantity is very natural from the analytic point of view and the resulting problem permits the use of tools from harmonic analysis such as a Littlewood-Paley decomposition. The relationship with conformal transformations becomes apparent when the domain is regarded as the boundary of a half-plane. The energy is invariant under the Möbius transformations that map the half-plane onto itself.

The functional has the disadvantage, however, that it depends on an embedding of the target manifold in a Euclidean space. Suppose that N is a smooth, compact Riemannian manifold without boundary. Then we always have an isometric embedding of N in a Euclidean space $\mathbb{R}^n$ by the theorem of Nash [9], and this embedding has proved a very useful tool in the theory of harmonic maps. Likewise, in this paper, we assume that N is embedded isometrically in $\mathbb{R}^n$ in order to study a variant of the harmonic map problem. But since we regard the ambient space merely as a tool, we prefer an intrinsically defined energy.

We define and analyse a functional that depends only on the geometry of the target and still enjoys the same type of conformal invariance for one-dimensional domains. The resulting theory requires that certain quantities normally studied in a linear context are replaced by nonlinear counterparts. In particular, the first variation of the functional is given by a Dirichlet-to-Neumann map for the

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harmonic map equation. Nevertheless, it is remarkably easy to derive some basic properties of the corresponding variational problem, since they can be reduced in a natural way to known facts about harmonic maps.

1 An intrinsic variational problem

For $m \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^m$ be an open set. We consider maps $u : \Omega \to N$. Define the half-space $\mathbb{R}^{m+1}_+ = \mathbb{R}^m \times (0, \infty)$. The functional studied by Da Lio and Rivière can be expressed in terms of the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ and the Dirichlet energy

$$E(v) = \frac{1}{2} \int_{\mathbb{R}^{m+1}_+} |\nabla v|^2 \, dx.$$

Let $T : \dot{H}^1(\mathbb{R}^{m+1}_+; \mathbb{R}^n) \to \dot{H}^{1/2}(\Omega; \mathbb{R}^n)$ be the trace operator with respect to the subset $\Omega \times \{0\}$ of the boundary. Then the functional

$$L(u) = \inf \left\{ E(v) : \ v \in \dot{H}^1(\mathbb{R}^{m+1}_+; \mathbb{R}^n), \ Tv = u \right\}$$

is well-defined on $\dot{H}^{1/2}(\Omega; \mathbb{R}^n)$. In the case $\Omega = \mathbb{R}^m$, it coincides up to a constant with the expression

$$\int_{\mathbb{R}^m} |\Delta^{1/4} u|^2 \, dx$$

defined in terms of Fourier multipliers. Critical points of L under the constraint that u takes values on N almost everywhere were studied in the aforementioned paper [2].

For the reasons given previously, we want to replace L by a different functional. Define

$$\dot{H}^1(\mathbb{R}^{m+1}_+; N) = \left\{ v \in \dot{H}^1(\mathbb{R}^{m+1}_+; \mathbb{R}^n) : \ v(x) \in N \text{ for almost every } x \in \mathbb{R}^{m+1}_+ \right\}$$

and let $\dot{H}^{1/2}(\Omega; N) = T(\dot{H}^1(\mathbb{R}^{m+1}_+; N))$ be its image under the trace operator. For $u \in \dot{H}^{1/2}(\Omega; N)$, let

$$I(u) = \inf \left\{ E(v) : \ v \in \dot{H}^1(\mathbb{R}^{m+1}_+; N), \ Tv = u \right\}.$$

If $\Omega = \mathbb{R}$, then the functional I is invariant with respect to group of all Möbius transformations that map $\mathbb{R}^2_+$ to itself, since the Dirichlet energy is invariant under conformal transformations of the domain.

We want to study critical points of I. First we need to examine variations of a map in $\dot{H}^{1/2}(\Omega; N)$, and to this end we observe that there exists a smooth nearest point projection $\pi_N : U \to N$, defined on a neighbourhood $U \subset \mathbb{R}^n$ of N. For $u \in \dot{H}^{1/2}(\Omega; N)$ and $\phi \in C_0^\infty(\Omega; \mathbb{R}^n)$, we consider the function

$$t \mapsto I(\pi_N \circ (u + t\phi)).$$

It is readily seen that this is well-defined at least in a neighbourhood of $t = 0$. Is this function differentiable at 0, and if so, what is its derivative?

Since I is defined as an infimum of quantities that are well understood, it is not difficult to identify a 'superdifferential', expressed in terms of minimizers of the Dirichlet energy E.

Definition 1.1. *Let $v \in \dot{H}^1(\mathbb{R}^{m+1}_+; N)$. We say that v is a minimal harmonic map if $E(v) \leq E(w)$ for all $w \in \dot{H}^1(\mathbb{R}^{m+1}_+; N)$ with $Tw = Tv$.*

Every $u \in \dot{H}^{1/2}(\Omega; N)$ has a minimal harmonic map $v \in \dot{H}^1(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$. This map satisfies the Euler-Lagrange equation

$$\Delta v + A(v)(\nabla v, \nabla v) = 0 \quad \text{in } \mathbb{R}^{m+1}_+$$

weakly. Here A is the second fundamental form of the embedding $N \subset \mathbb{R}^n$ and we use the shorthand notation

$$A(v)(\nabla v, \nabla v) = \sum_{\alpha=1}^{m} A(v)\left(\frac{\partial v}{\partial x^\alpha}, \frac{\partial v}{\partial x^\alpha}\right).$$

Given a function $\phi \in C_0^\infty(\Omega; \mathbb{R}^n)$, consider a $\psi \in C_0^\infty(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ with $\psi(x', 0) = \phi(x')$ for $x' \in \Omega$. Then the integral

$$\int_{\mathbb{R}^{m+1}_+} \left(\langle A(v)(\nabla v, \nabla v), \psi\rangle - \langle \nabla v, \nabla \psi\rangle\right) \, dx \tag{1}$$

depends on ϕ but not on its extension ψ. If v is sufficiently smooth up to the boundary, then we have of course

$$\int_{\mathbb{R}^{m+1}_+} \left(\langle A(v)(\nabla v, \nabla v), \psi\rangle - \langle \nabla v, \nabla \psi\rangle\right) \, dx = \int_\Omega \left\langle \frac{\partial v}{\partial x^{m+1}}(x', 0), \phi(x')\right\rangle \, dx'.$$

In general expression (1) gives rise to a distribution on Ω. Motivated by the above observation, we denote it by $\partial_{m+1} v$. That is,

$$\partial_{m+1} v(\phi) = \int_{\mathbb{R}^{m+1}_+} \left(\langle A(v)(\nabla v, \nabla v), \psi\rangle - \langle \nabla v, \nabla \psi\rangle\right) \, dx.$$

The superdifferential mentioned previously is then given by $-\partial_{m+1} v$.

Proposition 1.1. *Suppose that $u \in \dot{H}^{1/2}(\Omega; N)$, and that $v \in \dot{H}^1(\mathbb{R}^{m+1}_+; N)$ is a minimal harmonic map with $Tv = u$. Then for every $\phi \in C_0^\infty(\Omega; \mathbb{R}^n)$,*

$$I(\pi_N \circ (u + t\phi)) \leq I(u) - t\partial_{m+1} v(\phi) + o(|t|)$$

as $t \to 0$.

Proof. Let $u_t = \pi_N \circ (u + t\phi)$. For an extension $\psi \in C_0^\infty(\mathbb{R}^{m+1}; \mathbb{R}^n)$ of ϕ, define $v_t = \pi_N \circ (v + t\psi)$. Then $Tv_t = u_t$; hence $I(u_t) \leq E(v_t)$, with equality at $t = 0$. We compute

$$\left.\frac{d}{dt}\right|_{t=0} E(v_t) = -\partial_{m+1} v(\phi),$$

and the claim follows immediately. □

If the derivative

$$\left.\frac{d}{dt}\right|_{t=0} I(\pi_N \circ (u + t\phi))$$

exits, then it follows from this proposition that it is $-\partial_{m+1} v(\phi)$ for any minimal harmonic map $v \in \dot{H}^1(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$. In particular, this quantity is then independent of the choice of v, and we can interpret it as the value of a Dirichlet-to-Neumann mapping at u for the harmonic map problem.

Definition 1.2. *Let $\mathcal{D}$ be the set of all $u \in \dot{H}^{1/2}(\Omega; N)$ such that there exists a distribution $\lambda \in (C_0^\infty(\Omega; \mathbb{R}^n))^*$ satisfying $\lambda = -\partial_{m+1} v$ for every minimal harmonic map $v \in \dot{H}^1(\mathbb{R}_+^{m+1}; N)$ with $Tv = u$. In this case, set $\Lambda u = \lambda$, defining thus a map $\Lambda : \mathcal{D} \to (C_0^\infty(\Omega; \mathbb{R}^n))^*$.*

It turns out that Λu describes the first variation of I whenever it exists.

Theorem 1.1. *Let $u \in \dot{H}^{1/2}(\Omega; N)$. If $u \in \mathcal{D}$, then*

$$\left.\frac{d}{dt}\right|_{t=0} I(\pi_N \circ (u + t\phi)) = \Lambda u(\phi)$$

for all $\phi \in C_0^\infty(\Omega; \mathbb{R}^n)$. If $u \notin \mathcal{D}$, then there exists a $\phi \in C_0^\infty(\Omega; \mathbb{R}^n)$ such that the function $t \mapsto I(\pi_N \circ (u + t\phi))$ is not differentiable at 0.

The second part of this theorem follows of course from the previous observations. The first part is a little more difficult. We give a proof in the next section.

The result reduces the study of the first variation of I to questions about the Dirichlet-to-Neumann map Λ. The question is in particular whether a given $u \in \dot{H}^{1/2}(\Omega; N)$ is in the domain $\mathcal{D}$ of Λ. Obviously this is the case if the minimal harmonic map with these boundary data is unique. Uniqueness of harmonic maps has been studied by Struwe [14] and the author of this paper [8]. In both papers, the domain is assumed to be a unit ball, but the methods are not restricted to this situation, and conceivably, they give conditions that imply $u \in \mathcal{D}$. If $m = 1$, then it is in fact quite easy to see that u gives rise to a unique minimal harmonic map if $I(u)$ is sufficiently small. Other uniqueness results exits [7, 6], but under assumptions that are perhaps not so natural in the situation studied here.

On the other hand, there are clearly situations where $\dot{H}^{1/2}(\Omega; N) \backslash \mathcal{D}$ is nonempty. For example, consider the unit disk $D \subset \mathbb{R}^2$ with boundary S^1 and the unit sphere $S^2 \subset \mathbb{R}^3$ as the target manifold. The boundary data $u(x) = (x, 0)$, which map S^1 to the equator of S^2, give rise to two minimal harmonic maps on D, covering the upper and the lower hemisphere, respectively. Furthermore, these maps have different Neumann data. We can transform the disk conformally into $\mathbb{R}_+^2$, thus we draw similar conclusions for the corresponding map in $\dot{H}^{1/2}(\mathbb{R}; N)$.

If we study only minimizers, however, then this discussion is irrelevant.

Corollary 1.1. *If $u \in \dot{H}^{1/2}(\Omega; N)$ minimizes I in the sense that $I(\pi_N \circ (u + \phi)) \geq I(u)$ for all $\phi \in C_0^\infty(\Omega; \mathbb{R}^n)$, then $u \in \mathcal{D}$ and $\Lambda u = 0$.*

Proof. In this case, we have $I(\pi_N \circ (u + t\phi)) \geq I(u)$, and the claim follows from Proposition 1.1 and Theorem 1.1. □

Finally, we study the regularity of critical points of I. This turns out to be easier than the regularity theory for L in the work of Da Lio and Rivière [2], provided that we have $u \in \mathcal{D}$. We have in fact not only a regularity result, but a Liouville theorem as well. We first consider the one-dimensional case.

Theorem 1.2. *Suppose that $m = 1$ and $u \in \mathcal{D}$ satisfies $\Lambda u = 0$. Then $u \in C^\infty(\Omega; N)$. If $\Omega = \mathbb{R}$, then u is constant.*

Proof. Let $v \in \dot{H}^1(\mathbb{R}^2_+; N)$ be a minimal harmonic map with $Tv = u$. Extend v to $\mathbb{R}^2$ by $v(x_1, -x_2) = v(x_1, x_2)$. This makes v a weakly harmonic map on $\Omega \times \mathbb{R}$ and thus it is smooth by the results of Hélein [4, 5].

Now suppose that $\Omega = \mathbb{R}$. According to the removable singularity theorem of Sacks and Uhlenbeck [11], the composition of v with the stereographic projection can be extended to a smooth harmonic map on S^2. Every harmonic map on S^2 is conformal [3, Sect. 10], hence it follows that $\frac{\partial v}{\partial x^1} = 0$ on $\mathbb{R} \times \{0\}$. In other words, u is constant. $\square$

In higher dimensions, we need an additional stationarity condition to prove regularity, analogously to the regularity theory for harmonic maps.

Definition 1.3. *Let $u \in \mathcal{D}$ with $\Lambda u = 0$. We say that u is stationary if for every $\xi \in C_0^\infty(\Omega; \mathbb{R}^m)$,*

$$\left.\frac{d}{dt}\right|_{t=0} I(u(x + t\xi(x))) = 0. \tag{2}$$

Theorem 1.3. *Suppose that $u \in \mathcal{D}$ with $\Lambda u = 0$ is stationary. Then there exists a closed set $\Sigma \subset \Omega$ of vanishing $(m-1)$-dimensional Hausdorff measure, such that $u \in C^\infty(\Omega \backslash \Sigma; N)$. If $\Omega = \mathbb{R}^m$, then u is constant.*

Proof. The case $m = 1$ has already been treated, so we assume $m \geq 2$. Let $v \in \dot{H}^1(\mathbb{R}^{m+1}_+; N)$ be a minimal harmonic map with $Tv = u$. Consider the even extension to $\mathbb{R}^{m+1}$ as before. If we have $\zeta \in C_0^\infty(\mathbb{R}^{m+1}; \mathbb{R}^{m+1})$ with $\zeta(\,\cdot\,, 0) \in C_0^\infty(\Omega; \mathbb{R}^m \times \{0\})$, then we see that

$$\left.\frac{d}{dt}\right|_{t=0} E(v(x + t\zeta(x))) = 0 \tag{3}$$

with the same arguments as in the proof of Proposition 1.1. This is sufficient to prove the well-known monotonicity formula for harmonic maps [10] in balls $B_r(x_0) \subset \Omega \times \mathbb{R}$ with centre $x_0 \in \Omega \times \{0\}$ and also in balls contained entirely in $\mathbb{R}^{m+1}_+$. The arguments of Bethuel [1] now apply and give the required regularity.

If $\Omega = \mathbb{R}^m$, then we have the Pohozaev identity

$$\frac{m-1}{r} \int_{B_r(0)} |\nabla v|^2 \, dx = \int_{\partial B_r(0)} \left(|\nabla v|^2 - 2 \left| \frac{x}{|x|} \cdot \nabla v \right|^2 \right) d\sigma$$

for every $r > 0$, which also follows from (3) with standard arguments. Integrating with respect to r over an interval $(R, 2R)$, we obtain

$$(m-1) \int_{B_{2R}(0)} f_R(|x|) |\nabla v|^2 \, dx = \int_{B_{2R}(0) \backslash B_R(0)} \left(|\nabla v|^2 - 2 \left| \frac{x}{|x|} \cdot \nabla v \right|^2 \right) dx$$

for the function $f_R(\rho) = \log 2 + \min\{\log(R/\rho), 0\}$. For $R \to \infty$, the right-hand side tends to 0, as $v \in \dot{H}^1(\mathbb{R}^{m+1}; N)$. Hence v is constant. $\square$

2 Proof of Theorem 1.1

As pointed out previously, the second statement of Theorem 1.1 is a consequence of Proposition 1.1. In order to prove the first statement, we now examine the nearest point projection π_N more closely.

For $y \in N$, the derivative $D\pi_N(y)$ is the orthogonal projection onto the tangent space T_yN. The Hessian $D^2\pi_N(y)$ is related to the second fundamental form as follows. Let $X, Y \in T_yN$ and $\nu \perp T_yN$. Then

$$D^2\pi_N(y)(X, Y) = -A(y)(X, Y),$$

and $D^2\pi_N(y)(X, \nu) \in T_yN$ is the tangent vector characterized by

$$\left\langle Y, D^2\pi_N(y)(X, \nu)\right\rangle = -\left\langle A(y)(X, Y), \nu\right\rangle.$$

These are standard facts and a proof can be found in a book by Simon [13, Sect. 2.12.3].

For a fixed $\eta \in \mathbb{R}^n$, consider the mapping $y \mapsto \pi_N(y + \eta)$ (which comes into play when we study a map $\pi_N \circ (u + t\phi)$ as in the previous section). If $|\eta|$ is sufficiently small, then the inverse mapping is approximately $y \mapsto \pi_N(y - \eta)$. In order to study how good this approximation is, we define the map $\Phi_\eta : N \to N$ with

$$\Phi_\eta(y) = \pi_N(\pi_N(y + \eta) - \eta).$$

Obviously $\Phi_0 = \mathrm{id}_N$. Because $\Phi_\eta(y)$ is smooth in y and η, we see that $D\Phi_\eta(y)$ has full rank when $|\eta|$ is sufficiently small. Using the inverse function theorem and the compactness of N, we conclude that Φ_η is a diffeomorphism when $|\eta|$ is small enough. Let $\Psi_\eta = \Phi_\eta^{-1}$ whenever the inverse exists.

In the following, we also have to differentiate with respect to η. We use the notation D_η or D_y to indicate the variable of differentiation if necessary. For $X \in \mathbb{R}^n$, we have

$$D_\eta\Phi_\eta(y)X = D\pi_N(\pi_N(y + \eta) - \eta)(D\pi_N(y + \eta)X - X).$$

In particular $D_\eta\Phi_0(y)X = 0$. Hence there exists a constant C_1, dependent only on N, such that for all $y \in N$ and for $|\eta|$ small enough, we have

$$|\Phi_\eta(y) - y| \le C_1|\eta|^2 \quad \text{and} \quad |D_\eta\Phi_\eta(y)| \le C_1|\eta|,$$

and furthermore,

$$|\Psi_\eta(y) - y| \le C_1|\eta|^2 \quad \text{and} \quad |D_\eta\Psi_\eta(y)| \le C_1|\eta|.$$

Now let $Y \in T_yN$. Then

$$D_y\Phi_\eta(y)Y = D\pi_N(\pi_N(y + \eta) - \eta)D\pi_N(y + \eta)Y$$

and

$$\begin{aligned} D_\eta(D_y\Phi_\eta(y)Y)X = {} & D^2\pi_N(\pi_N(y + \eta) - \eta)(D\pi_N(y + \eta)Y, D\pi_N(y + \eta)X - X) \\ & + D\pi_N(\pi_N(y + \eta) - \eta)D^2\pi_N(y + \eta)(X, Y). \end{aligned}$$

Hence

$$D_\eta(D_y\Phi_0(y)Y)X = D^2\pi_N(y)(Y, D\pi_N(y)X - X) + D\pi_N(y)D^2\pi_N(y)(X, Y) = 0.$$

We conclude that there is another constant C_2, dependent only on N, such that

$$|D_y\Phi_\eta(y)Y - Y| \le C_2|\eta|^2|Y| \quad \text{and} \quad |D_y\Psi_\eta(y)Y - Y| \le C_2|\eta|^2|Y|.$$

Now we consider the situation of Theorem 1.1. We assume that $u \in \mathcal{D}$, that is, there exists a distribution Λu such that for every minimal harmonic map $v \in \dot{H}^1(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$, we have $\partial_{m+1} v = -\Lambda u$.

Fix v with these properties. Let $\phi \in C_0^\infty(\Omega; \mathbb{R}^n)$ and choose an extension $\psi \in C_0^\infty(\mathbb{R}^{m+1}; \mathbb{R}^n)$. For $t \in (-1, 1)$ with $|t|$ sufficiently small, define

$$u_t = \pi_N \circ (u + t\phi) \quad \text{and} \quad v_t = \pi_N \circ (v + t\psi).$$

Furthermore, let $w_t \in \dot{H}^1(\mathbb{R}^{m+1}_+; N)$ be a minimal harmonic map with $Tw_t = u_t$. In the following, we use the symbols $C_3, C_4, \dots$ to denote constants that depend only on N and the choice of ψ, without mentioning the dependence again. First, it is easy to see that we have a constant C_3 such that

$$E(v_t) \leq I(u) + C_3 |t| (I(u) + 1), \tag{4}$$

and obviously $I(u_t) = E(w_t) \leq E(v_t)$. Next we define

$$\tilde{u}_t = \pi_N \circ (u_t - t\phi) \quad \text{and} \quad \tilde{w}_t = \pi_N \circ (w_t - t\psi).$$

We have

$$\begin{aligned} \nabla \tilde{w}_t &= D\pi_N(w_t - t\psi)(\nabla w_t - t\nabla\psi) \\ &= \nabla w_t + (D\pi_N(w_t - t\psi) - D\pi_N(w_t))\nabla w_t - tD\pi_N(w_t - t\psi)\nabla\psi. \end{aligned}$$

There is a constant C_4 such that

$$\left|(D\pi_N(w_t - t\psi) - D\pi_N(w_t))\nabla w_t + tD^2\pi_N(w_t)(\nabla w_t, \psi)\right| \leq C_4 t^2 |\nabla w_t|.$$

Hence

$$|\nabla \tilde{w}_t|^2 \leq |\nabla w_t|^2 - 2t \langle \nabla w_t, \nabla\psi \rangle + 2t \langle A(w_t)(\nabla w_t, \nabla w_t), \psi \rangle + C_5 t^2 (|\nabla w_t|^2 + 2)$$

and

$$E(\tilde{w}_t) \leq I(u_t) + t\partial_{m+1} w_t(\phi) + C_5 t^2 (E(w_t) + 1).$$

Of course we have $\tilde{u}_t(x) = \Phi_{t\phi(x)}(u(x))$. If we define $\hat{w}_t(x) = \Psi_{t\psi(x)}(\tilde{w}_t(x))$, then $T\hat{w}_t = u$. Note that

$$\nabla \hat{w}_t(x) = D_y \Psi_{t\psi(x)}(\tilde{w}_t(x)) \nabla \tilde{w}_t(x) + tD_\eta \Psi_{t\psi(x)}(\tilde{w}_t(x)) \nabla\psi(x).$$

Using the estimates for the derivatives of Ψ_η, we obtain a constant C_6 such that

$$|\nabla \hat{w}_t - \nabla \tilde{w}_t| \leq C_6 t^2 (|\nabla \tilde{w}_t| + 1).$$

That is,

$$E(\hat{w}_t) \leq E(\tilde{w}_t) + C_7 t^2 (E(\tilde{w}_t) + 1).$$

Combining these inequalities, we find a constant C_8 with

$$I(u) \leq I(u_t) + t\partial_{m+1} w_t(\phi) + C_8 t^2 (I(u) + 1). \tag{5}$$

Choose a sequence $t_k \to 0$. As $\{w_{t_k}\}_{k\in\mathbb{N}}$ is bounded in $\dot{H}^1(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$, there exists a subsequence t_{k_ℓ} such that $w_{t_{k_\ell}} \rightharpoonup w$ weakly in $\dot{H}^1(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ for some $w \in \dot{H}^1(\mathbb{R}^{m+1}_+; N)$ with $Tw = u$. Since v is a minimal harmonic map, we have

$$E(w) \geq E(v) \geq \limsup_{\ell\to\infty} E(w_{t_{k_\ell}})$$

(the second inequality due to (4)), hence the convergence is even strong in $\dot{H}^1(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$. Furthermore, w is a minimal harmonic map as well. It follows that $\partial_{m+1}w$ is well-defined and

$$\partial_{m+1}w(\phi) = \lim_{\ell\to\infty} \partial_{m+1}w_{t_{k_\ell}}(\phi). \tag{6}$$

Since $u \in \mathcal{D}$, we have $\partial_{m+1}w(\phi) = -\Lambda u(\phi)$. That is, the limit in (6) does not depend on the subsequence t_{k_ℓ}, and therefore we have in fact

$$\Lambda u(\phi) = -\lim_{t\to 0} \partial_{m+1}w_t(\phi).$$

Inequality (5) now implies

$$I(u_t) \geq I(u) + t\Lambda u(\phi) + o(|t|)$$

as $t \to 0$. Together with Proposition 1.1, this yield the desired identity. The proof of Theorem 1.1 is now complete.

3 A constrained version

Suppose now that $\Gamma \subset N$ is a smooth, closed submanifold. Consider the set

$$\dot{H}^{1/2}_N(\Omega;\Gamma) = \left\{ u \in \dot{H}^{1/2}(\Omega;N) : \ u(x') \in \Gamma \text{ for almost every } x' \in \Omega \right\}$$

and the restriction of I to $\dot{H}^{1/2}_N(\Omega;\Gamma)$. We think of Γ as a constraint and we want to examine critical points of I under this constraint.

First we have to define what we mean by a critical point of this variational problem. It is no longer appropriate to consider the quantity $\pi_N \circ (u + t\phi)$ for $\phi \in C^\infty_0(\Omega;\mathbb{R}^n)$, as this does not preserve the constraint in general. Instead we use a weak version of the formal Euler-Lagrange equation, which can be represented in the form

$$\Lambda u \perp T_u N \quad \text{in } \Omega. \tag{7}$$

Consider the space $\mathcal{G} = H^1(\mathbb{R}^{m+1};\mathbb{R}^n) \cap L^\infty(\mathbb{R}^{m+1};\mathbb{R}^n)$ with the topology generated by all sets of the form $U \cap V$, where U is open in $H^1(\mathbb{R}^{m+1};\mathbb{R}^n)$ and V is weakly* open in $L^\infty(\mathbb{R}^{m+1};\mathbb{R}^n)$ (regarded as the dual space of $L^1(\mathbb{R}^{m+1};\mathbb{R}^n)$). Let $\mathcal{F} = T(\mathcal{G})$ be its image under the trace operator T, equipped with the quotient topology. We are interested in the closure of $C^\infty_0(\Omega;\mathbb{R}^n)$ in $\mathcal{F}$, which we denote by $\mathcal{F}_0$.

One of the reasons for choosing this (somewhat unwieldy) topology is that it makes $C^\infty_0(\mathbb{R}^{m+1};\mathbb{R}^n)$ dense in $\mathcal{G}$, in fact even in the sequential sense. Hence $C^\infty_0(\mathbb{R}^m;\mathbb{R}^n)$ is dense in $\mathcal{F}$ and we have the following localization property: if $\phi \in \mathcal{F}$ and $\chi \in C^\infty_0(\Omega)$, then $\chi\phi \in \mathcal{F}_0$.

For a harmonic map $v \in \dot{H}^1(\mathbb{R}^{m+1}_+;N)$, the functional

$$\psi \mapsto \int_{\mathbb{R}^{m+1}_+} \left(\langle A(v)(\nabla v, \nabla v), \psi\rangle - \langle \nabla v, \nabla\psi\rangle\right) dx$$

is continuous on $\mathcal{G}$. It is constant on the fibres of T, hence it induces a unique continuous extension of $\partial_{m+1}v$ to $\mathcal{F}_0$. We conclude that for $u \in \mathcal{D}$, there exists a unique continuous extension of Λu to $\mathcal{F}_0$, which we also denote by Λu.

Definition 3.1. *We say that* $u \in \mathcal{D} \cap \dot{H}^{1/2}_N(\Omega;\Gamma)$ *is a critical point of* I *under the constraint* Γ *if for every* $\phi \in \mathcal{F}_0$ *the following holds: if* $\phi(x') \in T_{u(x')}\Gamma$ *for almost every* $x' \in \Omega$, *then* $\Lambda u(\phi) = 0$. *If condition* (2) *holds as well, then we say that* u *is stationary.*

Again regularity can be obtained from known results about harmonic maps.

Theorem 3.1. *If* $m = 1$ *and* $u \in \mathcal{D} \cap \dot{H}^{1/2}_N(\Omega;\Gamma)$ *is a critical point of* I *under the constraint* Γ, *then* $u \in C^\infty(\Omega;\Gamma)$.

Proof. Let $v \in \dot{H}^1(\mathbb{R}^2_+;N)$ be a minimal harmonic map with $Tv = u$. Then v essentially satisfies the conditions used by Scheven [12] to derive regularity. There is a potential conflict with Scheven's assumptions near the boundary $\partial\Omega$ (which are not given explicitly), but the arguments in this paper are local and we can always use a cut-off function to avoid problems. The desired regularity follows directly. □

Theorem 3.2. *Let* $m \geq 2$ *and suppose that* $u \in \mathcal{D} \cap \dot{H}^{1/2}_N(\Omega;\Gamma)$ *is a stationary critical point of* I *under the constraint* Γ. *Then there exists a closed set* $\Sigma \subset \Omega$ *of vanishing* $(m-1)$*-dimensional Hausdorff measure, such that* $u \in C^\infty(\Omega\backslash\Sigma;\Gamma)$. *If* $\Omega = \mathbb{R}^m$, *then* u *is constant.*

Proof. The regularity follows from the results of the same paper [12] again. The proof of the second statement is the same as in the case of Theorem 1.3. □

We do not have a Liouville theorem for $m = 1$. In fact there is the following simple counterexample. By the conformal invariance, we can replace the domain by the unit circle S^1 and $\mathbb{R}^2_+$ by the unit disk D. We wish to consider the submanifold $\Gamma = S^1$ of the target $\mathbb{R}^2$, but since compactness of N is required, we regard S^1 as a submanifold of a torus $N = \mathbb{R}^2/r\mathbb{Z}^2$ instead for some $r > 2$. Every harmonic map $v \in H^1(D;N)$ can be lifted to $\mathbb{R}^2$, giving rise to a solution of the Laplace equation. Hence v is determined uniquely by its boundary data, and we conclude that $\mathcal{D} = H^{1/2}(S^1;N)$. For the identity map $S^1 \to S^1$, we can now verify the Euler-Lagrange equation (7), and it follows that this is a critical point of I under the constraint S^1.

In the case $N = \mathbb{R}^n/r\mathbb{Z}^n$, the constrained variational problem for I is essentially the same as the extrinsic problem for the functional L studied by Da Lio and Rivière [2], with Γ taking the role of N. In particular, Theorem 3.1 corresponds to the main result of their paper. Hence Theorems 3.1 and 3.2 can be regarded as generalizations of the theory. Furthermore, the arguments are less involved (owing in part to the fact that most of the work has been done elsewhere). The proof of Da Lio and Rivière may, however, give more insight into the problem, as it uses the special structure of the Euler-Lagrange equation explicitly.

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